CURVATURE TENSOR UNDER THE RICCI-HARMONIC FLOW

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ABSTRACT. We prove that if the Ricci curvature is uniformly bounded under the Ricci-Harmonic flow for all times $t \in [0, T)$, then the curvature tensor has to be uniformly bounded as well.

1. INTRODUCTION

The Ricci flow has been a useful tool in the study of geometry problem. It played an important part in the proof of the Poincare Conjecture and Geometric Conjecture. Since then, geometric flows attract more attention. In [5], B.List studied an extended Ricci flow

$$\begin{cases} \frac{\partial}{\partial t} g_{ij} = -2R_{ij} + \alpha \phi_i \phi_j, & t \in [0, T); \\ \frac{\partial}{\partial t} \phi = \Delta \phi, & t \in [0, T). \end{cases}$$

In his dissertation, B.List said this flow has some applications in general relativity. The nonlocal collapse and monotonicity of some entropy have been proved. Some gradient estimate and local existence of the flow are obtained. As long as the Riemannian curvature tensor is bounded, the flow will exists and the gradient and Hession of ϕ are also bounded, [5]. R.Muller also introduced a new geometric flow, called Ricci-Harmonic flow,

$$\begin{cases} \frac{\partial}{\partial t} g_{ij} = -2R_{ij} + \alpha \phi_i \phi_j, & t \in [0, T); \\ \frac{\partial}{\partial t} \phi = \tau_{g(t)} \phi, & t \in [0, T). \end{cases}$$

where $\phi : (M, g(t)) \to (N, h)$ is a map between two Riemannian manifold, $\tau_g \phi = trace \nabla d\phi$. The extended Ricci flow of B.List is a special case of Ricci-Harmonic flow. Similar to the extended Ricci flow, this flow will also exist if the Riemannian curvature tensor is bounded.

On the other hand, there are many results about controlling the Riemannian curvature tensor under Ricci flow. In [9], by a blow up argument, N.Susem showed that Ricci curvature uniformly bounded on $M \times [0, T)$, where $T < \infty$ is enough to control the norm of Riemannian curvature tensor on closed manifold. Then L.Ma and L.Cheng improveed the result to the Ricci flow on complete noncompact manifold. R.Ye [12] and B.Wang [13], by different arguments, proved that the norm of the Riemannian curvature tensor can be controlled by the bound of $\|Rm\|_{\frac{n+2}{2}} = (\int_0^T \int_M |Rm|^{\frac{n+2}{2}} d\mu dt)^{\frac{2}{n+2}}$. In this paper, we will prove that if the Ricci curvature is uniformly bounded, the flow can continue.

The following are the main results of this paper.

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Theorem 1. *Let $(M, g(t), \phi(t)), t \in [0, T)$ be a solution to the Ricci Harmonic flow on a Riemannian manifold. Suppose $T < \infty$ and Ricci curvature is uniformly bounded under the flow. Then the Riemannian curvature tensor $|Rm|$ stays unoformly bounded under the flow.*

In the following, we will prove 1, Since the Extended Ricci flow is a special case of the Ricci Harmonic flow. Note that the proof doesn't depend on the compact of the manifold. We just take a closed manifold for simple.

If the manifold is compact, we also have the following result.

Theorem 2. *Let $(M, g(t), \phi(t)), t \in [0, T)$, where $T < \infty$, be a solution to the Ricci-Harmonic flow on a closed manifold satisfying*

$$\|R\|_{\frac{n+2}{2}} = (\int_0^T \int_M |R|^{\frac{n+2}{2}} d\mu dt) < \infty,$$

and

$$\|W\|_{\frac{n+2}{2}} = (\int_0^T \int_M |W|^{\frac{n+2}{2}} d\mu dt) < \infty,$$

Then $\sup_{M \times [0,T)} \|Rm\| < \infty$.

In the following, we denote $S_{ij} = R_{ij} - \alpha \phi_i \phi_j$, $S = R - \alpha |\nabla \phi|^2$.

2. Preliminary

Lemma 3. *Under the same condition of the theorem 1, we have*

$$|\nabla \phi|(t) < C, t \in [0, T).$$

where C is independent of t.

Proof. Since Ric curvature is uniformly bounded along the flow at $t \in [0, T)$, the Scale curvature is also uniformly bounded. From [4], we have

$$(\partial_t - \Delta) S = 2|S_{ij}|^2 + 2\alpha |\tau_g \phi(t)|^2.$$

So the minimum of $S(t)$ is nondecreasing along the flow.

$$S(t) = R - \alpha |\nabla \phi|^2 \geq C.$$

Since R is uniformly bounded, $|\nabla \phi(t)|^2 \leq C$, where C is independent of t.

□

Since Ric curvature and $|\nabla \phi(t)|^2$ is uniformly bouhnded, we will control the variation of the distance along the Extended Ricci flow.

Lemma 4. [5],[14] *Under the same condition of theorem 1. For all $\delta > 0$, there exists an $\eta > 0$, such that if $|t - t_0| < \eta, t, t_0 \in [0, T)$, then*

$$|d_{g(t)}(q, q^{'}) - d_{g(t_0)}(q, q^{'})| \leq \delta d_{g(t_0)}(q, q^{'})$$

for all $q, q^{'} \in M$.

Proof.

$$\begin{aligned}|\log g(t)(V,V)-\log g(t_0)(V,V)| &= |\int_{t_0}^{t}\frac{\partial}{\partial t}\log g(s)(V,V)ds| \\ &= |\int_{t_0}^{t}\frac{(-2Ric(V,V)+2\alpha<\nabla\phi,V>^2)}{|V|^2_{g(s)}}ds| \\ &\leq \int_{t_0}^{t}\frac{2C|V|^2_{g_s}+2\alpha|\nabla\phi|^2_{g(s)}|V|^2_{g(s)}}{|V|^2_{g_s}}ds \\ &\leq C(t-t_0)\end{aligned}$$

So we have

$$e^{-C|t-t_0|}|V|^2_{g(t_0)}\leq|V|^2_{g(t)}\leq e^{C|t-t_0|}|V|^2_{g(t_0)}$$

Suppose γ_1 is a minimal geodesic from p to $p^{'}$ with respect to metric $g(t_0)$.

$$d_{g(t)}(p,p^{'})\leq\int|\dot{\gamma_1}|_{g(t)}(s)ds\leq\int e^{C|t-t_0|/2}|\dot{\gamma_1}|_{g(t_0)}(s)ds=e^{C|t-t_0|/2}d_{g(t_0)}(p,p^{'}).$$

Similarly, Let γ_2 be a minimal geodesic from p to $p^{'}$ with respect to metric $g(t)$. Then

$$d_{g(t)}(p,p^{'})=\int|\dot{\gamma_2}|_{g(t)}(s)ds\geq\int e^{-C|t-t_0|/2}|\dot{\gamma_2}|_{g(t_0)}(s)ds\geq e^{-C|t-t_0|/2}d_{g(t_0)}(p,p^{'}).$$

$$|d_{g(t)}(p,p^{'})-d_{g(t_0)}(p,p^{'})|\leq(e^{C|t-t_0|/2}-1)d_{g_{t_0}}(p,p^{'}).$$

The lemma will be satisfied if we take $\eta=\frac{2\ln(\delta+1)}{C}$. □

Corollary 5. *Under the condition of theorem 1. For all $\rho>0$,*

$$B_{g(t)}(0,r(t)\rho)\subset B_{g(0)}(0,\rho),$$
$$B_{g(0)}(0,r(t)\rho)\subset B_{g(t)}(0,\rho),$$

where $r(t)=e^{-Ct/2}$.

Proof. Suppose $x\in B_{g(t)}(o,r(t)\rho)$. Then $r(t)\rho\geq d_{g(t)}(o,x)\geq e^{-Ct/2}d_{g(0)}(o,x)$. So $d_{g(0)}(o,x)\leq\rho$. Similarly for $B_{g(0)}(0,r(t)\rho)\subset B_{g(t)}(0,\rho)$. □

Lemma 6. *Under the same condition of theorem 1. For all $\epsilon>0$, there exists an $\delta(\epsilon,C)>0$, such that if $t\in[t_0,t_0+\delta]$ then*

$$Vol_{g(t)}B_{g(t)}(x,r)\geq(1-\epsilon)^{\frac{n}{2}}Vol_{g(t_0)}B_{g(t_0)}(x,\frac{r}{1+\epsilon}).$$

Proof. By the corollary 5, for all $\epsilon>0$, there exists $\delta_1>0$, such that if $t\in[t_0,t_0+\delta_1]$, we have

$$B_{g(t_0)}(x,\frac{r}{1+\epsilon})\subset B_{g(t)}(x,r).$$

Since the Ricci curvature is uniformly bounded in $[0,T)$, by lemma 3, we have $|S|<C$ is uniformly bounded.

$$\frac{d}{dt}\int_{B_{g(t_0)}(x,\frac{r}{1+\epsilon})}dv=\int_{B_{g(t_0)}(x,\frac{r}{1+\epsilon})}-Sdv$$

i.e.

$$|\frac{d}{dt}Vol_{g(t)}B_{g(t_0)}(x,\frac{r}{1+\epsilon})|<CVol_{g(t)}B_{g(t_0)}(x,\frac{r}{1+\epsilon}).$$

□

We have

$$Vol_{g(t)}B_{g(t_0)}(x, \frac{r}{1+\epsilon}) \geq e^{-C(t-t_0)}Vol_{g(t_0)}B_{g(t_0)}(x, \frac{r}{1+\epsilon})$$

There exists a $\delta_2 > 0$, such that $e^{-C(t-t_0)} \geq (1-\epsilon)^{\frac{n}{2}}$, if $t \in [t_0, t_0 + \delta_2]$.

Take $\delta = \min\{\delta_1, \delta_2\}$, for $t \in [t_0, t_0 + \delta]$

$$Vol_{g(t)}B_{g(t)}(x, r) \geq (1-\epsilon)^{\frac{n}{2}}Vol_{g(t_0)}B_{g(t_0)}(x, \frac{r}{1+\epsilon}).$$

Lemma 7. *Let $\{(M_i^n, g_i(t), \phi_i(t), x_i)\}_{i=1}^{\infty}, t \in [0, T]$ be a sequence of the Ricci-Harmonic flow on complete manifolds such that $\sup_{M_i \times [0,T]} |Rm(g_i(t))|_{g_i(t)} \leq 1$, $\sup_{x \in M} |\phi_i(x, 0)| < C$ where C is independent of i. Let $\psi_i = \exp_{x_i, g_i(0)}$ be the exponential map with respect to metric $g_i(0)$ and $B(o_i, \frac{\pi}{2}) \subset T_{x_i}M_i$ equipped with metric $\tilde{g}_i(t) \doteq \psi_i^* g_i(t), \tilde{\phi}_i = \phi_i \circ \psi_i$. Then $(B(o_i, \frac{\pi}{2}), \tilde{g}_i(t), \tilde{\phi}_i(t), o_i)$ subconverges to a Ricci-Harmonic flow $(B(o, \frac{\pi}{2}), \tilde{g}(t), \tilde{\phi}, o)$ in C^∞ sense, where $B(o, \frac{\pi}{2}) \subset R^n$ equipped with metric $\tilde{g}(t)$.*

Proof. Since $\tilde{g}_i(t) \doteq \psi_i^* g_i(t)$, we have $\sup_{B(o_i, \frac{\pi}{2}) \times [0,T]} |Rm(\tilde{g}_i(t))|_{\tilde{g}_i(t)} \leq 1$. On $B(o_i, \frac{\pi}{2})$, we have $inj(o_i, \tilde{g}_i(0)) \geq \frac{\pi}{2}$ [7]. Then the lemma follows from the compactness theorem in [14],[5]. □

3. Blow up analysis

Proof. By the Maximum principle, we have

$$\inf_{x \in M} \phi(x, 0) \leq \phi(x, t) \leq \sup_{x \in M} \phi(x, 0)$$

for $t \in [0, T)$. In the following, we consider the pull back metrics on tangent spaces. Suppose the Ricci-Harmonic flow blows up at a finite time T. That means there exist sequences $t_i \to T$ and $p_i \in M$ such that

$$Q_i = |Rm|(p_i, t_i) \geq C^{-1} \max_{M \times [0, t_i]} |Rm|(x, t).$$

where $C > 1$ and $Q_i \to \infty$ as $i \to \infty$. By lemma 7, the scaled flow $(B(o_i, \frac{\pi}{2C}), g_i(t), \phi_i(t))$ will converge to a complete ancient solution $(B(o, \frac{\pi}{2C}), \overline{g}(t), \overline{\phi})$ of the Ricci-Harmonic flow, where $g_i(t) = Q_i g(t_i + \frac{t}{Q_i}), \phi_i(t) = \phi(t_i + \frac{t}{Q_i})$. Since $S(t) = R(t) - \alpha|\nabla\phi(t)|^2$ is uniformly bounded, $S_i(t) = \frac{1}{Q_i}(R(g(t_i + \frac{t}{Q_i})) - \alpha|\nabla\phi|^2(g(t_i + \frac{t}{Q_i})))$. As $i \to \infty$, $S_i(t) \to 0$. Our ancient solution $(B(o, \frac{\pi}{2C}), \overline{g}(t), \overline{\phi})$ has $S = 0$. The evolution of S is

$$(\partial_t - \Delta)S = 2|S_{ij}|^2 + 2\alpha|\Delta\phi|^2.$$

This imply that $|S_{ij}| = 0, |\Delta\phi| = 0$. i.e. $Ric = \alpha\nabla\phi \bigotimes \nabla\phi \geq 0$.

In the following, we will analyze the volume of the limit $(B(o, \frac{\pi}{2C}), \overline{g}(t), \overline{\phi})$.

Take $r > 0$, since the convergence is C^∞, the volume is convergence,

$$\frac{VolB(p, r)}{r^n} = \lim_{i \to \infty} \frac{Vol_i B_i(p_i, r)}{r^n}$$

where the volume and the ball $B(p, r)$ on the LHS are considered in metric $\overline{g}$, while the RHS are considered in metric $g_i(0) = Q_i g(t_i)$. So

$$\frac{VolB(p, r)}{r^n} = \lim_{i \to \infty} \frac{Vol_{g(t_i)}B_{g(t_i)}(p_i, rQ_i^{-1/2})}{(Q_i^{-1/2}r)^n}$$

For $\forall\epsilon > 0$, there exists $N > 0$, such that if $i > N$, we have

$$\frac{VolB(p,r)}{r^n} \geq \frac{Vol_{g(t_i)}B_{g(t_i)}(p_i, rQ_i^{-1/2})}{(Q_i^{-1/2}r)^n} - \frac{\epsilon}{2}.$$

For such $\epsilon > 0$, there exists a $\delta > 0$ from lemma 6. We take $t_0 > T - \delta$. For i sufficiently large, $0 < t_i - t_0 < \delta$,

$$\frac{VolB(p,r)}{r^n} \geq (1-\epsilon)^{n/2}\frac{Vol_{g(t_0)}B_{g(t_0)}(p_i, \frac{1}{1+\epsilon}rQ_i^{-1/2})}{(Q_i^{-1/2}r)^n} - \frac{\epsilon}{2}.$$

Note that volume has the expansion within the injective radius of p(see [2] theorem 3.98),

$$VolB(p,r) = \omega_n r^n(1 - \frac{R(p)}{6(n+2)}r^2 + o(r^2)).$$

Let $i \to \infty$, we have

$$\frac{VolB(p,r)}{r^n} \geq \omega_n\frac{(1-\epsilon)^{n/2}}{(1+\epsilon)^n} - \frac{\epsilon}{2}.$$

Let $\epsilon \to 0$, we have

$$\frac{VolB(p,r)}{r^n} \geq \omega_n.$$

Since $Ric \geq 0$, by the Bishop-Gromov volume comparison, we have (N,g) is a flat manifold, which is contradict to the $|Rm|(p,0) = 1$.

$\square$

Proof. From Theorem 8.6 in [5], since M is compact, the limit solution is in fact a ancient solution to Ricci flow. The proof is the similar to the proof in [8]. $\square$

Remark 8. The blow up can also be applied to the complete noncompact Riemannian manifold as in [8].

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